A Crossover on SAW in random environment

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Introduction of SAW

Two point function:

$$G_p(x) = \sum_{n=0}^{\infty} c_n(x)\; p^n$$
$$= \sum_{n=0}^{\infty} \left(\sum_{\substack{\omega : o \to x \\ |\omega| = n}} \prod_{i=1}^{n} D(\omega_{i-1}, \omega_i) \prod_{0 \le s < t \le n} (1 - \delta_{\omega_s, \omega_t}) \right) \; p^{|\omega|}$$

Susceptibility:

$$\chi(p) = \sum_{x \in \mathbb{Z}^d} G_p(x) = \sum_{n=0}^{\infty} c_n \; p^n$$

Connective constant:

$$\lim_{n \to \infty} (c_n)^{\frac{1}{n}} =: \mu \in (0, \infty)$$

Critical point and Asymptotic behaviour of SAW

Fact 1. **Critical point**:

$$\chi(p) = \sum_{n=0}^{\infty} \left((c_n)^{\frac{1}{n}} \; p \right)^n \sim \sum_{n=0}^{\infty} (\mu \, p)^n \quad \Rightarrow \quad p_c = \frac{1}{\mu}$$

Fact 2. **Critical exponent**: $\chi(p) \asymp \dfrac{1}{(p - p_c)^\gamma}$

$$\gamma = \begin{cases} 1 & d = 1 \\ \frac{43}{32} & d = 2 \\ 1.162\ldots & d = 3 \\ 1 \text{ with } \log \text{ corr.} & d = 4 \\ 1 & d > 4 \end{cases}$$

▷ **SAW behaves like simple RW** when **higher dimensions** $d > 4$

→ **Mean field behaviour** of SAW

Motivation for SAW on random conductors

Statistical mechanics

Self-avoiding walk

- Phase transition
- Critical phenomenon

Want to know

Critical behaviour

- Critical point
- Critical exponent

Disorder system

Random environment: $\boldsymbol{X} = \{X_b\}_{b \subset \mathbb{B}^d}$

Motivation

I want to know the effect of the random environment !!

SAW in random environment

Susceptibility : generating function of $c_{\beta,\boldsymbol{X}}(x;n)$

$$\chi_{h,\beta,\boldsymbol{X}}(x) = \sum_{n=0}^{\infty} \left(\sum_{\omega\in\Omega_n(x)} e^{-\beta \sum_{j=1}^{n} X_{b_j}} \right) e^{-nh} = \sum_{n=0}^{\infty} c_{\beta,\boldsymbol{X}}(x;n)\, e^{-nh}$$

- $\Omega_n(x)$: the set of n-step SAW path from x
- $p = e^{-h}$: **fugacity** ($h \in \mathbb{R}$: **chemical potential/energy cost**)
- $\beta \geq 0$: strength of randomness
- $\boldsymbol{X} = (X_b)$: set of random conductors
  - attached to each edge/bond
  - i.i.d. and bounded
  - probability law $\mathbb{P}$ (distributed according to ν)
  - finite moment: $\lambda_\beta = \mathbb{E}[e^{-\beta X_b}] < \infty$

Critical point for SAW in random environment

Question

What is the critical point (= radius of convergence)?

1. Define formally

$$h_c^{\mathfrak{q}}(\beta, \boldsymbol{X} : x) := \inf\{h \in \mathbb{R} \colon \chi_{h,\beta,\boldsymbol{X}}(x) < \infty\}$$

2. **Open:** Analogy of homogeneous case

$$h_c^{\mathfrak{q}}(\beta, \boldsymbol{X} : x) = \lim_{n\to\infty} \frac{1}{n} \log c_{\beta,\boldsymbol{X}}(x; n)$$

Proposition 1

The critical point:

$$h_c^{\mathfrak{q}}(\beta, \boldsymbol{X} : x) = \limsup_{n\to\infty} \frac{1}{n} \log\, c_{\beta,\boldsymbol{X}}(x; n)$$

Theorem 1 [C.- and Sakai]

For $d \geq 1$ and $\beta \geq 0$, the critical point $h_c^{\mathsf{q}}(\beta, \boldsymbol{X})$ is almost surely an x-independent constant and

$$\log \mu - \beta\, \mathbb{E}[X_b] \leq h_c^{\mathsf{q}}(\beta, \boldsymbol{X}) \leq h_c^{\mathsf{a}}(\beta) = \log \mu + \mathbb{E}\left[e^{-\beta X_b}\right]$$

- **Homogeneous model** $\Rightarrow \quad h_c = \log \mu$
- $d = 1 \quad \Rightarrow \quad h_c^{\mathsf{q}}(\beta, \boldsymbol{X}) = -\beta\, \mathbb{E}[X_b] \quad (c_{\beta, \boldsymbol{X}}(x; n) = 2)$
- The 2nd moment estimate:

$$\Rightarrow \quad h_c^{\mathsf{q}}(\beta, \boldsymbol{X}) \leq h_c^{\mathsf{a}}(\beta).$$

- The fractional moment estimate:

$$\Rightarrow \quad h_c^{\mathsf{q}}(\beta, \boldsymbol{X}) < h_c^{\mathsf{a}}(\beta).$$

Crossover: strong disorder vs weak disorder

β is large $\Rightarrow$ Randomness from the environment affects much

There should exist a threshold β_c whether randomness affects or not

$\Downarrow$

Crossover

strong disorder (large β) vs **weak disorder** (small β)

Conjecture

On $\mathbb{Z}^d$, there exists a $\beta_c \geq 0$ such that

$$h_c^{\mathsf{q}}(\beta, \boldsymbol{X}) \begin{cases} = h_c^{\mathsf{a}}(\beta) & \text{if} \quad \beta < \beta_c \quad (\textbf{weak disorder}) \\ < h_c^{\mathsf{a}}(\beta) & \text{if} \quad \beta > \beta_c \quad (\textbf{strong disorder}) \end{cases}$$

Crossover in $\mathbb{Z}^d$

Let

$$Z_n := \frac{c_n(\beta, \boldsymbol{X})}{\mathbb{E}[c_n(\beta, \boldsymbol{X})]}$$

Note that

$$Z_n = \frac{1}{c_n \, \lambda_\beta^n} \; \exp\left\{ n \left(\frac{1}{n} \log c_n(\beta, \boldsymbol{X}) \right) \right\}$$

$$= \exp\left\{ n \left(\frac{1}{n} \log(c_n \, \lambda_\beta^n) - \frac{1}{n} \log c_n(\beta, \boldsymbol{X}) \right) \right\}$$

Observation

If $\lim_{n\to\infty} Z_n(\beta, \boldsymbol{X}; x) =: Z_\infty$ exists, then

$$h_c^{\mathsf{q}}(\beta, \boldsymbol{X}) < h_c^{\mathsf{a}}(\beta) \quad \Leftrightarrow \quad Z_\infty = 0$$

$$h_c^{\mathsf{q}}(\beta, \boldsymbol{X}) = h_c^{\mathsf{a}}(\beta) \quad \Leftrightarrow \quad Z_\infty \in (0, \infty)$$

Results and Conjectures on $\mathbb{Z}^d$

Crossover

- in one dimension, there is **no crossover**
- in two dimension, there is **no crossover**
- in three dimension, there will be **no crossover** (Open)
- in more than four dimensions, there will be **crossover** (Open)

Z_n is related to **end-to-end distance**.

- Z_n is of order 1, the trajectories of SAW in random environment behaves like homogeneous case
- Z_n decays exponentially fast in n, each trajectory tends to concentrate in a region where the path obtains favourable conductors/potentials.

On tree: Analogy from directed polymer models

Proposition 2

On Cayley tree structure, Z_n is a martingale.

- with memory 1 or 2 $\Rightarrow$ Z_n is a **martingale**.
- on $\mathbb{Z}^d$ $\Rightarrow$ Z_n is a **supermartingale**.

- By martingale convergence, there exists a limit

$$\lim_{n\to\infty} Z_n =: Z_\infty \quad \in [0, \infty)$$

- By Kolmogorov's 0-1 law, we have $\mathbb{P}(Z_\infty = 0) = 0$ or 1

Theorem 2

On d-aray tree, for $d \geq 3$, there exists a $\beta_c \geq 0$ such that

$$h_c^{\mathsf{q}}(\beta, \boldsymbol{X}) \begin{cases} = h_c^{\mathsf{a}}(\beta) & \text{if} \quad \beta < \beta_c \quad (\textbf{weak disorder}) \\ < h_c^{\mathsf{a}}(\beta) & \text{if} \quad \beta > \beta_c \quad (\textbf{strong disorder}) \end{cases}$$

感謝大家的聆聽

Thank you very much for your attention